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Central Limit Theorems for Percolation Models

J. Theodore Cox[1] **and Geoffrey Grimmett**[2]

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Let $p \neq 1/2$ be the open-bond probability in Broadbent and Hammersley's percolation model on the square lattice. Let W_x be the cluster of sites connected to x by open paths, and let $\{\gamma(n)\}$ be any sequence of circuits with interiors $|\mathring{\gamma}(n)| \to \infty$. It is shown that for certain sequences of functions $\{f_n\}$, $S_n = \sum_{x \in \mathring{\gamma}(n)} f_n(W_x)$ converges in distribution to the standard normal law when properly normalized. This result answers a problem posed by Kunz and Souillard, proving that the number S_n of sites inside $\gamma(n)$ which are connected by open paths to $\gamma(n)$ is approximately normal for large circuits $\gamma(n)$.

KEY WORDS: Percolation; asymptotic normality; circuits; semi-invariants.

1. INTRODUCTION

The percolation model of Broadbent and Hammersley[2] has received considerable recent attention in both the mathematics and physics literature (see Refs. 5–8, 10, 14–16). In this paper we will verify a conjecture of Kunz and Souillard (Section 6 in Ref. 10) and prove a general central limit theorem.

We will consider only the bond percolation model on the square lattice, although some of our methods should work for other models. Let $\mathbb{Z}_2$ be the set of points in the plane with integer coefficients, and for $x, y \in \mathbb{Z}_2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$, write $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$. The points of $\mathbb{Z}_2$ will be called *sites*, and the line segments joining sites x and y with $d(x, y) = 1$ will be called *bonds*. The origin is the site (0, 0), denoted by 0. Each bond is declared *open* with probability p or *closed* with probability $1 - p$ independently of all other bonds. We will assume throughout that $0 < p < 1$.

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[1] Syracuse University, Syracuse, New York.
[2] University of Bristol, Bristol, England.

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A *path* from x to y is an alternating sequence of sites and bonds of the form $x_0, v_1, x_1, v_2, \ldots, v_n, x_n$, where $x_0 = x$, $x_n = y$, $d(x_i, x_{i+1}) = 1$, and v_i is the bond joining x_{i-1} and x_i. A *circuit* γ is a path $x_0, v_1, x_1, \ldots, v_n, x_n$ such that $x_0, x_1, \ldots, x_{n-1}$ are distinct and $x_0 = x_n$. Thus a circuit is a special type of a simple closed curve. Let $\mathring{\gamma}$ be the set of sites strictly *inside* γ and let $\partial\mathring{\gamma}$ be the *inside boundary*, $\partial\mathring{\gamma} = \{x \in \mathring{\gamma} |$ there exists $y \in \gamma$ with $d(x, y) = 1\}$. The notation $x \sim y$ means there is a path from x to y with *all bonds open*, and for any $A \subset \mathbb{Z}_2$, $x \sim A$ means $x \sim y$ for some $y \in A$. The number of sites in A is denoted $|A|$.

The *open cluster* W_x at x is defined to be the set of all sites y such that $x \sim y$. If the four bonds of x are all closed, W_x is empty. For any $A \subset \mathbb{Z}_2$ let $W_A = \cup_{x \in A} W_x$. The usual interpretation of W_x is that it represents the set of sites which are "wetted" by placing a fluid source at x and allowing fluid to flow only along open bonds. It is now known (see Ref. 8) that $P(|W_x| = \infty)$ is zero for $p \leqslant 1/2$ and strictly positive for $p > 1/2$.

Given a circuit γ we can alter our viewpoint by putting fluid sources at each site of γ and asking how many sites *inside* γ are "wetted." In Ref. 10 Kunz and Souillard conjectured that the number of such sites should be approximately normal for large γ. We verify this conjecture with the following theorem.

Theorem 1. Assume $p \neq 1/2$ and define $\xi_\gamma(x) = 1_{\{W_x \cap \gamma \neq \varnothing\}}$ and $S_\gamma = \sum_{x \in \mathring{\gamma}} \xi_\gamma(x)$. Then there are finite, nonzero constants $c_1(p)$, $c_2(p)$ such that for all circuits γ,

$$p < 1/2 \quad \text{implies} \quad c_1(p)|\partial\mathring{\gamma}| \leqslant ES_\gamma, \operatorname{Var} S_\gamma \leqslant c_2(p)|\partial\mathring{\gamma}| \tag{1.1}$$

and

$$p > 1/2 \quad \text{implies} \quad c_1(p)|\mathring{\gamma}| \leqslant ES_\gamma, \operatorname{Var} S_\gamma \leqslant c_2(p)|\mathring{\gamma}| \tag{1.2}$$

Furthermore, if $\{\gamma(n)\}$ is any sequence of circuits with $|\mathring{\gamma}(n)| \to \infty$, then $(S_{\gamma(n)} - ES_{\gamma(n)})/(\operatorname{Var} S_{\gamma(n)})^{1/2}$ converges in distribution to the standard normal law.

The estimates on ES_γ and $\operatorname{Var} S_\gamma$ indicate an essential qualitative difference in the behavior of S_γ for p above or below the critical value of $1/2$. By making "regularity" assumptions on the circuits $\{\gamma(n)\}$ it is possible to obtain more precise results. In particular, if γ_n is the boundary of the square $[0, n] \times [0, n]$, then

$$p < 1/2 \quad \text{implies} \begin{cases} ES_{\gamma(n)} = 4(n-1)\lambda + O(\ln n) \\ \operatorname{Var} S_{\gamma(n)} = 4(n-1)\Lambda_1 + O(\ln n) \end{cases} \tag{1.3}$$

and

$$p > 1/2 \quad \text{implies} \begin{cases} ES_{\gamma(n)} = (n-1)^2 p_\infty + 4(n-1)\lambda + O(\ln n) \\ \operatorname{Var} S_{\gamma(n)} = (n-1)^2 \Lambda_2 + O(n \ln n) \end{cases} \tag{1.4}$$

where the constants λ, p_∞, Λ_1, Λ_2 are defined in Section 5.

A more general central limit theorem which applies to sums of functions of the open clusters can be formulated as follows. A function f is said to be *increasing* (*decreasing*) on the subsets of $\mathbb{Z}_2$ if $f(W_1) \leqslant f(W_2)$ $(\geqslant)$ whenever $W_1 \subset W_2$. Let $\mathcal{F}$ be the set of (finite) real valued functions f defined on the connected subsets of $\mathbb{Z}_2$ which are either increasing or decreasing, and are *constant* on infinite sets [i.e., $f(W_1) = f(W_2)$ if $|W_1| = |W_2| = +\infty$].

Theorem 2. Assume $p \neq 1/2$ and $\{\gamma(n)\}$ is a sequence of circuits with $|\mathring{\gamma}(n)| \to \infty$. Assume $\{f_n\}$ is a sequence of functions satisfying the following:

$$f_n \in \mathcal{F} \quad \text{for each } n \tag{1.5}$$

$$\sup_n \max_{x \in \mathring{\gamma}(n)} E|f_n(W_x)|^k = C_k < \infty \qquad \text{for each } k = 1, 2, \ldots \tag{1.6}$$

$$\inf_n \min_{x \in \mathring{\gamma}(n)} \operatorname{Var}[f_n(W_x)] = \sigma^2 > 0 \tag{1.7}$$

If $S_{\gamma(n)} = \sum_{x \in \mathring{\gamma}(n)} f_n(W_x)$, then $(S_{\gamma(n)} - ES_{\gamma(n)})/(\operatorname{Var} S_{\gamma(n)})^{1/2}$ converges in distribution to the standard normal law.

Several remarks are in order here. It will be shown in Section 4 that Theorem 2 covers the $p > 1/2$ case of Theorem 1, but not the $p < 1/2$ case. The problem is condition (1.7). It will be seen in Section 2 [see (2.1)] that (1.6) is not overly restrictive. For the case $p > 1/2$, the number $S_\gamma = \sum_{x \in \mathring{\gamma}} \xi_\gamma(x)$ of sites inside γ which are joined to γ by open paths may be divided into two components

$$S_\gamma = S_\gamma^I + S_\gamma^F$$

where

$$S_\gamma^I = \sum_{x \in \mathring{\gamma}} 1_{\{|W_x| = +\infty\}}, \quad S_\gamma^F = \sum_{x \in \mathring{\gamma}} \eta_\gamma(x),$$

and $\eta_\gamma(x) = 1_{\{|W_x| < \infty, W_x \cap \gamma \neq \varnothing\}}$. The functions $f_n(W_x) = 1_{\{|W_x| = +\infty\}}$ satisfy the hypotheses of Theorem 2 for any sequence $\{\gamma(n)\}$, and we obtain a slightly more general version of a result of Grimmett (see Ref. 6), which is a central limit theorem for the number of sites inside $\gamma(n)$ which are "connected to infinity." The second component S_γ^F of S_γ is also asymptotically normally distributed; in fact it follows easily from Lemma 1 and the proof of (1.1) that there exist constants $c_1'(p)$ and $c_2'(p)$ such that

$$c_1'(p)|\partial\mathring{\gamma}| \leqslant ES_\gamma^F, \operatorname{Var} S_\gamma^F \leqslant c_2'(p)|\partial\mathring{\gamma}|$$

and that $(S_{\gamma(n)}^F - ES_{\gamma(n)}^F)/(\operatorname{Var} S_{\gamma(n)}^F)^{1/2}$ converges to the standard normal law.

It is also possible to consider $f_n(W_x) = |W_{x,n}|^{-1} 1_{\{|W_{x,n}|>0\}}$ where $W_{x,n}$ is the set of sites in $\mathring{\gamma}(n)$ joined to x by open paths within $\mathring{\gamma}(n)$. Although f_n is not monotone, it is still possible to prove asymptotic normality for $\sum_{x\in\mathring{\gamma}(n)} f_n(W_x)$, the number of open clusters in $\mathring{\gamma}(n)$.

It should be noted here that Theorem 2 above is similar to Theorem (3.1) in Neaderhouser's paper,[13] except that strict regularity requirements for the circuits $\gamma(n)$ are imposed there. Although the approach used in Ref. 13 may possibly be modified to allow arbitrary circuits $\gamma(n)$, we feel that Malyšev's method of Ref. 12 using the method of moments and semi-invariants is simpler, and works easily for both Theorems 1 and 2. Malyšev's technique is also used in Refs. 1 and 9.

2. THE BASIC INEQUALITIES

Lemma 1. If $p \neq 1/2$ then there are finite nonzero constants α and β depending only on p such that

$$P(|W_0| < \infty \quad \text{and there exists } y \in W_0, d(0, y) \geqslant m) \leqslant \alpha e^{-\beta m}$$

Proof. For $p < 1/2$ W_0 is finite with probability 1, and part of Theorem 2 in Ref. 8 states that there exists some $\beta_1(p)$, $0 < \beta_1(p) < \infty$, such that P (there exists $y \in W_0$, $d(0, y) \geqslant m) \leqslant 2e^{-\beta_1(p)m}$. For $p > 1/2$ we turn to the dual-lattice technique, explained in Ref. 15. If $|W_0| < \infty$ and there exists $y \in W_0$ with $d(0, y) \geqslant m$, then there must exist some circuit of closed bonds in the dual-lattice containing W_0 (and hence the origin) with length at least m. Such a circuit must contain at least one of the dual-lattice sites $x_k^* = (k + 1/2, 1/2), k \geqslant 0$. Therefore, $P(|W_0| < \infty$ and there exists $y \in W_0, d(0, y) \geqslant m) \leqslant$

$$\sum_{k=0}^{\infty} P(\text{there is a path in the dual-lattice containing } x_k^* \text{ with length } \geqslant \max(m,k))$$

$$\leqslant \sum_{k=0}^{m} 2e^{-\beta_1(1-p)m} + \sum_{k=m+1}^{\infty} 2e^{-\beta_1(1-p)k}$$

$$\leqslant \alpha e^{-\beta m}$$

for an appropriate choice of α, β depending only on p. ∎

Corollary. If $p \neq 1/2$ and A is a finite subset of $\mathbb{Z}_2$, then $P(|W_A| < \infty$ and there exists $y \in W_A$ with $d(x, y) \geqslant m$ for all $x \in A) \leqslant \alpha |A| e^{-\beta m}$.

Throughout the rest of the paper α and β will be the constants defined in Lemma 1. It follows from Lemma 1 that condition (1.6) of Theorem 2 is satisfied for a sequence $\{f_n\}$ if there is a function g such that $|g(\infty)| < \infty$, $\sup_n |f_n(W_x)| \leqslant g(|W_x|)$ and

$$\sum_{n=1}^{\infty} g(n)^k \exp(-\beta_1 n^{1/2}) < \infty \qquad \text{for all } k = 1, 2, \ldots \tag{2.1}$$

Remark. Since H. Kesten has recently shown (personal communication) that $P(|W_0| \geqslant n) \leqslant \alpha' e^{-\beta' n}$ for $p < 1/2$ where α', β' depend only on p, (2.1) can be improved accordingly.

We can now state and prove a result which will show that $f_n(W_x)$ is more or less "independent" of $f_n(W_y)$ if x and y are "far apart." Let $d(x, A) = \min\{d(x, y) \mid y \in A\}$ and $d(A, B) = \min\{d(x, B) \mid x \in A\}$.

Lemma 2. Assume $p \neq 1/2$ and $\{\gamma_n\}$ is a sequence of circuits with $|\mathring{\gamma}(n)| \to \infty$. Let $\{f_n\}$ be a sequence of functions which satisfy conditions (1.5) and (1.6) and the additional requirement that $f_n(W_x) = 0$ if $|W_x| = +\infty$. For finite sets $A \subset \mathbb{Z}_2$ let $\rho_n(A) = \Pi_{x \in A} f_n(W_x)$. Then for all finite sets $A, B \subset \mathbb{Z}_2$ there exists a finite constant c_3 depending only on p, $|A|$, $|B|$, and the numbers C_k in (1.6) such that for all n,

$$|E\rho_n(A)\rho_n(B) - E\rho_n(A)E\rho_n(B)| \leqslant c_3 e^{-\beta d(A,B)/4}$$

Proof. We first observe that repeated application of Hölder's inequality to $E|\rho_n(A)|$ and condition (1.6) show that $E|\rho_n(A)|$ is bounded above by a quantity which depends only on p, $|A|$, and the numbers C_k in (1.6). Let $m = d(A, B)$ and let $\Omega_A = \{d(x, A) \leqslant m/2$ for all $x \in W_A\}$ and $\Omega_B = \{d(x, B) \leqslant m/2$ for all $x \in W_B\}$. Then Ω_A and Ω_B are *independent*, and writing $E(X; G)$ for $E(X 1_G)$, $E(\rho_n(A)\rho_n(B); \Omega_A \cap \Omega_B) = E(\rho_n(A); \Omega_A)$ $E(\rho_n(B); \Omega_B)$. Thus

$$\begin{aligned}
\big|E\rho_n(A)\rho_n(B) - E\rho_n(A)E\rho_n(B)\big| &\leqslant \big|E(\rho_n(A)\rho_n(B); \Omega_A^c \cup \Omega_B^c)\big| \\
&\quad + \big|E(\rho_n(A); \Omega_A)E(\rho_n(B); \Omega_B) - E\rho_n(A)E\rho_n(B)\big| \\
&\leqslant E(|\rho_n(A)\rho_n(B)|; \Omega_A^c \cup \Omega_B^c) + E|\rho_n(A)|E(|\rho_n(B)|; \Omega_B^c) \\
&\quad + E|\rho_n(B)|E(|\rho_n(A)|; \Omega_A^c)
\end{aligned}$$

Since $\rho_n(A) = 0$ if $|W_A| = +\infty$ and $\rho_n(B) = 0$ if $|W_B| = +\infty$, we can let $\tilde{\Omega}_A = \Omega_A^c \cap \{|W_A| < \infty\}$ and $\tilde{\Omega}_B = \Omega_B^c \cap \{|W_B| < \infty\}$ and bound the terms above by

$$E(|\rho_n(A)\rho_n(B)|;\tilde{\Omega}_A) + E(|\rho_n(A)\rho_n(B)|;\tilde{\Omega}_B) + E|\rho_n(A)|E(|\rho_n(B)|;\tilde{\Omega}_B) + E|\rho_n(B)|E(|\rho_n(A)|;\tilde{\Omega}_A)$$

Replacing $\tilde{\Omega}_A$ and $\tilde{\Omega}_B$ with their indicator functions and using Hölder's inequality we have

$$\begin{aligned}
|E\rho_n(A)\rho_n(B) - E\rho_n(A)E\rho_n(B)| &\leqslant \left[E\rho_n^2(A)\rho_n^2(B)\right]^{1/2}\left[(E1_{\tilde{\Omega}_A})^{1/2} + (E1_{\tilde{\Omega}_B})^{1/2}\right] \\
&\quad + E|\rho_n(A)|\left[E\rho_n^2(B)\right]^{1/2}(E1_{\tilde{\Omega}_B})^{1/2} \\
&\quad + E|\rho_n(B)|\left[E\rho_n^2(A)\right]^{1/2}(E1_{\tilde{\Omega}_A})^{1/2} \\
&\leqslant c_3 e^{-\beta m/4}
\end{aligned}$$

using the Corollary to Lemma 1 to estimate $E1_{\tilde{\Omega}_A}$ and $E1_{\tilde{\Omega}_B}$, where c_3 depends on p, $|A|$, $|B|$, and the C_k in (1.6). ■

3. PROOF OF THEOREM 2

We start by pointing out that it suffices to consider sequences $\{f_n\}$ such that $f_n(W_x) = 0$ if $|W_x| = +\infty$. This is because any $f_n \in \mathcal{F}$ can be written as $f_n = d_n + g_n$, where $d_n = f_n(\mathbb{Z}_2)$ and $g_n(W_x) = [f_n(W_x) - d_n]1_{\{|W_x|<\infty\}}$. Next we observe that the FKG inequalities (see Refs. 3, 4, and 16) imply that $Ef_n(W_x)f_n(W_y) - Ef_n(W_x)Ef_n(W_y) \geqslant 0$, so that

$$\begin{aligned}
\operatorname{Var} S_{\gamma(n)} &= \sum_{x\in\mathring{\gamma}(n)}\sum_{y\in\mathring{\gamma}(n)} Ef_n(W_x)f_n(W_y) - Ef_n(W_x)Ef_n(W_y) \\
&\geqslant \sum_{x\in\mathring{\gamma}(n)} \operatorname{Var} f_n(W_x) \\
&\geqslant \sigma^2|\mathring{\gamma}(n)|
\end{aligned}$$

by condition (1.7).

It is now necessary to recall certain facts about semi-invariants and Ursell functions, which can be found in Refs. 9, 11, and 12. The kth semi-invariant of a random variable X will be written $\nu_k(X)$; it is the coefficient of t^k in the expansion of $\ln Ee^{tX}$, and can be expressed in terms of the moments of X by

$$\nu_k(X) = \sum_{m=1}^{k}(-1)^{m-1}\frac{1}{m}\sum_{\substack{r_1,r_2,\ldots,r_m\geqslant 1\\ r_1+r_2+\cdots+r_m=k}}\frac{k!}{r_1!r_2!\cdots r_m!}EX^{r_1}EX^{r_2}\cdots EX^{r_m} \tag{3.1}$$

The Ursell function $\nu(X_1, X_2, \ldots, X_n)$ of n variables is

$$\nu(X_1, \ldots, X_n) = \sum_{s=1}^{n} (-1)^{s-1}(s-1)! \sum_{\underline{\underline{\pi}} : |\underline{\underline{\pi}}| = s} E(X_{i_1^1} X_{i_2^1} \ldots) \times E(X_{i_1^2} X_{i_2^2} \ldots) \cdots E(X_{i_1^s} X_{i_2^s} \ldots) \tag{3.2}$$

where the second sum is over all partitions $\underline{\underline{\pi}} = \{\{i_1^1, i_2^1, \ldots\}, \{i_1^2, i_2^2, \ldots\} \cdots \{i_1^s, i_2^s, \ldots\}\}$ of $\{1, 2, \ldots, n\}$ consisting of s members. It is known (see Refs. 9, 11, 12) that the semi-invariants of a sum can be written as

$$\nu_k\left(\sum_{i=1}^{n} X_i\right) = \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \nu(X_{i_1}, \ldots, X_{i_k}) \tag{3.3}$$

Since the moments are determined by the semi-invariants, to prove $Z_n = (S_{\gamma(n)} - ES_{\gamma(n)})/(\operatorname{Var} S_{\gamma(n)})^{1/2}$ converges in distribution to the standard normal law it suffices to prove $\nu_k(Z_n) \to 0$ if $k \neq 2$ and $\nu_2(Z_n) \to 1$ (the semi-invariants of the standard normal law). Note that $\nu_1(Z_n) = EZ_n = 0$ and $\nu_2(Z_n) = \operatorname{Var} Z_n = 1$.

For any finite set $A \subset \mathbb{Z}_2$ define $G_{k,m}(A)$ by

$$G_{k,m}(A) = \Big\{(x_1, x_2, \ldots, x_k) \,|\, \text{each } x_i \in A \quad \text{and the maximum over nontrivial partitions } \{\pi', \pi''\} \text{ of } \{1, 2, \ldots, k\} \text{ of } \min_{i \in \pi', j \in \pi''} d(x_i, x_j) \quad \text{is equal to } m\Big\} \tag{3.4}$$

A counting argument (see Ref. 9) shows that

$$|G_{k,m}(A)| \leqslant |A|(2m+1)^{2k}(k!)^2 \tag{3.5}$$

The last formula needed (see Refs. 9, 11, 12) expresses the Ursell functions in terms of moments. Let $\{\pi', \pi''\}$ be a nontrivial partition of $\{1, 2, \ldots, k\}$, fix $(x_1, x_2, \ldots, x_k) \in G_{k,m}$, and let $\rho_n(\pi) = \prod_{i \in \pi} f_n(W_{x_i})$ for any subset $\pi \subset \{1, 2, \ldots, k\}$. Then

$$\nu(f_n(W_{x_1}), f_n(W_{x_2}), \ldots, f_n(W_{x_k})) = \sum_{\substack{\underline{\underline{\pi}} = \{\pi_1, \pi_2, \pi_3, \ldots\} \\ \pi_1 \subset \pi', \pi_2 \subset \pi''}} \pm \big[E\rho_n(\pi_1)\rho_n(\pi_2) - E\rho_n(\pi_1)E\rho_n(\pi_2)\big] E\rho_n(\pi_3)E\rho_n(\pi_4) \cdots \tag{3.6}$$

where the sum is over nontrivial partitions $\underline{\underline{\pi}}$ of $\{1, 2, \ldots, k\}$ and the sign depends on $\underline{\underline{\pi}}$. For each $(x_1, x_2, \ldots, x_k) \in G_{k,m}$, by choosing the particular

$\{\pi', \pi''\}$ such that $\min_{i\in\pi', j\in\pi''} d(x_i, x_j) = m$, and using Lemma 2, we obtain.

$$|\nu(f_n(W_{x_1}), \ldots, f_n(W_{x_k}))| \leqslant c_3' \cdot \Gamma_k \cdot e^{-\beta m/4} \tag{3.7}$$

where c_3' is $\sup_n \max_{\pi\subset\{1,2,\ldots,k\}}[1 + E\rho_n^2(\pi)]^k$ and Γ_k is the number of partitions of $\{1, 2, \ldots, k\}$.

Using (3.7) in (3.3) we obtain

$$\begin{aligned}
\nu_k(Z_n) &\leqslant (\operatorname{Var} S_{\gamma(n)})^{-k/2} \sum_{m=0}^{\infty} \sum_{(x_1,\ldots,x_k)\in G_{k,m}(\mathring{\gamma}(n))} \nu(f_n(W_{x1}), \ldots, f_n(W_{x_k}))\\
&\leqslant \sigma^{-k}|\mathring{\gamma}(n)|^{-k/2} \sum_{m=0}^{\infty} |\mathring{\gamma}(n)|(2m+1)^k(k!)^2 c_3'\Gamma_k e^{-\beta m/4}\\
&= |\mathring{\gamma}(n)|^{1-k/2}\sigma^{-k}c_3'(k!)^2\Gamma_k \sum_{m=0}^{\infty}(2m+1)^k e^{-\beta m/4}\\
&\to 0
\end{aligned}$$

for $k \geqslant 3$.

4. PROOF OF THEOREM 1

We will consider only the case $p < 1/2$, since for $p > 1/2$,

$$\begin{aligned}
\operatorname{Var}\xi_\gamma(x) &= P(W_x \cap \gamma \neq \varnothing)P(W_x \cap \gamma = \varnothing)\\
&\geqslant P(|W_x| = +\infty)\cdot(1-p)^4
\end{aligned}$$

a bound which implies Theorem 2 applies. Throughout the remainder of this section we assume $p < 1/2$. We will first prove the variance estimates of (1.1), omitting the similar estimates on ES_γ. Since the FKG inequality implies $E\xi_\gamma(x)\xi_\gamma(y) - E\xi_\gamma(x)E\xi_\gamma(y) \geqslant 0$,

$$\begin{aligned}
\operatorname{Var} S_\gamma &= \sum_{x\in\mathring{\gamma}}\sum_{y\in\mathring{\gamma}} E\xi_\gamma(x)\xi_\gamma(y) - E\xi_\gamma(x)E\xi_\gamma(y)\\
&\geqslant \sum_{x\in\partial\mathring{\gamma}} E\xi_\gamma(x)[1 - E\xi_\gamma(x)]\\
&= \sum_{x\in\partial\mathring{\gamma}} P(W_x \cap \gamma \neq \varnothing)P(W_x \cap \gamma = \varnothing)
\end{aligned}$$

Since $x \in \partial\mathring{\gamma}$, $P(W_x \cap \gamma \neq \varnothing) \geqslant p$ and $P(W_x \cap \gamma = \varnothing) \geqslant (1-p)^4$, and $\operatorname{Var} S_\gamma \geqslant p(1-p)^4|\partial\mathring{\gamma}|$. To obtain an upper bound we introduce $R_i(\gamma) = \{x \in \mathring{\gamma} \mid d(x, \partial\mathring{\gamma}) = i\}$ and $\tilde{\xi}_\gamma(x) = 1 - \xi_\gamma(x)$. Note that $E\xi_\gamma(x)\xi_\gamma(y) - E\xi_\gamma(x)$ $E\xi_\gamma(y) = E\tilde{\xi}_\gamma(x)\tilde{\xi}_\gamma(y) - E\tilde{\xi}_\gamma(x)E\tilde{\xi}_\gamma(y)$. If we let $i_\gamma = \max\{i \mid R_i(\gamma) \neq \varnothing\}$, it is clear that $i_\gamma \leqslant |\partial\mathring{\gamma}|$.

$$
\begin{aligned}
\operatorname{Var} S_\gamma &= \sum_{i=1}^{i_\gamma} \sum_{x \in R_i(\gamma)} \sum_{y \in \mathring{\gamma}} E\xi_\gamma(x)\xi_\gamma(y) - E\xi_\gamma(x)E\xi_\gamma(y) \\
&= \sum_{i=1}^{i_\gamma} \sum_{x \in R_i(\gamma)} \sum_{\substack{y \in \mathring{\gamma} \\ d(x,y) \leqslant i}} E\xi_\gamma(x)\xi_\gamma(y) - E\xi_\gamma(x)E\xi_\gamma(y) \\
&\quad + \sum_{i=1}^{i_\gamma} \sum_{x \in R_i(\gamma)} \sum_{\substack{y \in \mathring{\gamma} \\ d(x,y) > i}} E\tilde{\xi}_\gamma(x)\tilde{\xi}_\gamma(y) - E\tilde{\xi}_\gamma(x)E\tilde{\xi}_\gamma(y) \\
&\leqslant \sum_{i=1}^{\infty} \sum_{x \in R_i(\gamma)} \sum_{\substack{y \in \mathring{\gamma} \\ d(x,y) \leqslant i}} P(W_x \cap \gamma \neq \emptyset) \\
&\quad + \sum_{i=1}^{\infty} \sum_{x \in R_i(\gamma)} \sum_{\substack{y \in \mathring{\gamma} \\ d(x,y) > i}} c_3 e^{-\beta d(x,y)/4}
\end{aligned}
$$

using Lemma 2, which applies because $\tilde{\xi}_\gamma(x) = 0$ if $|W_x| = +\infty$. We can now use the rather crude estimate $|R_i(\gamma)| \leqslant 4\sqrt{2}\, i|\partial\mathring{\gamma}|$, for $i \geqslant 1$, and Lemma 1 to bound the preceding terms by

$$
4\sqrt{2}\,|\partial\mathring{\gamma}| \left\{ \sum_{i=1}^{\infty} i(+2i+1)^2 \alpha e^{-\beta i} + \sum_{i=1}^{\infty} \sum_{\substack{y \in \mathbb{Z}_2 \\ d(0,y) \geqslant i}} i c_3 e^{-\beta d(0,y)/4} \right\}
$$

which proves (1.1).

To prove $S_{\gamma(n)}$ is approximately normal we will use the method of Section 3 after we show that only the terms $\xi_{\gamma(n)}(x)$ with x near $\partial\mathring{\gamma}(n)$ contribute significantly to $S_{\gamma(n)}$. To do this, define $T_K(\gamma) = \{x \in \mathring{\gamma} \mid d(x, \partial\mathring{\gamma}) \geqslant K\}$, where K will be chosen later. The terms of $S_{\gamma(n)}$ which come from $T_K(\gamma(n))$ have variance

$$
\begin{aligned}
&\operatorname{Var}\Bigl(\sum_{x \in T_K(\gamma(n))} \xi_\gamma(x) \Bigr) \\
&\quad = \sum_{x \in T_K(\gamma(n))} \sum_{y \in T_K(\gamma(n))} E\xi_{\gamma(n)}(x)\xi_{\gamma(n)}(y) - E\xi_{\gamma(n)}(x)E\xi_{\gamma(n)}(y) \\
&\quad \leqslant \sum_{x \in T_K(\gamma(n))} \sum_{y \in T_K(\gamma(n))} P(\text{there exists } z \in W_x, d(z,x) \geqslant K) \\
&\quad \leqslant |T_K(\gamma(n))|^2 \alpha e^{-\beta K}
\end{aligned}
$$

by Lemma 1. The additional crude estimates of $|T_K(\gamma(n))| \leqslant |\mathring{\gamma}(n)|^2$ and $|\mathring{\gamma}(n)| = \sum_{i=1}|R_i(\gamma(n))| \leqslant 4\sqrt{2}\,|\partial\mathring{\gamma}(n)|^3$, with the choice of $K = |\partial\mathring{\gamma}(n)|^{1/5}$, give

$$\operatorname{Var}\left(\sum_{x \in T_K(\gamma(n))} \xi_{\gamma(n)}(x)\right) \to 0$$

It suffices then to check $\nu_k(\sum_{x \in \mathring{\gamma}(n)\setminus T_K(\gamma(n))}\xi_{\gamma(n)}(x)/(\operatorname{Var} S_{\gamma(n)})^{1/2}) \to 0$ for $k \geqslant 3$. As in Section 3 we bound ν_k by

$$\begin{aligned}(\operatorname{Var} S_{\gamma(n)})^{-k/2} &\sum_{m=0}^{\infty} |\mathring{\gamma}(n)\setminus T_K(\gamma(n))|(2m+1)^k(k!)^2\Gamma_k c_3 e^{-\beta m/4} \\ &\leqslant \left(p(1-p)^4\right)^{-k/2}|\partial\mathring{\gamma}(n)|^{-k/2}|\mathring{\gamma}(n)\setminus T_K(\gamma(n))| \cdot c_3\Gamma_k(k!)^2 \\ &\sum_{m=0}^{\infty}(2m+1)^k e^{-\beta m/4} \to 0\end{aligned}$$

for $k \geqslant 3$, since with $K = |\partial\mathring{\gamma}(n)|^{1/5}$,

$$\begin{aligned}|\mathring{\gamma}(n)\setminus T_K(\mathring{\gamma}(n))| &\leqslant \sum_{i=0}^{K}|R_i(\gamma(n))| \leqslant |\partial\mathring{\gamma}(n)|\left(1 + \frac{4\sqrt{2}\,K(K+1)}{2}\right) \\ &= O\left(|\partial\mathring{\gamma}(n)|^{1+2/5}\right) \quad \blacksquare\end{aligned}$$

5. ESTIMATES FOR THE SQUARES

In this section $\gamma(n)$ will be the boundary of the square $[0,n] \times [0,n]$, and we will write S_n for $S_{\gamma(n)}$ and $\xi_n(x)$ for $\xi_{\gamma(n)}(x)$. The constants in (1.3) and (1.4) are

$$\begin{aligned}p_\infty &= P(|W_0| = +\infty) \\ \lambda &= \sum_{n=1}^{\infty} P(|W_0| < \infty \quad \text{and} \quad W_0 \cap H_n \neq \varnothing) \\ \Lambda_2 &= \sum_{y \in \mathbb{Z}_2} P(|W_0| = |W_y| = +\infty) - p_\infty^2 \qquad (5.1) \\ \Lambda_1 &= \sum_{n=1}^{\infty}\sum_{\substack{y \in \mathbb{Z}_2 \\ y_1 < n}} \{P(W_0 \cap H_n \neq \varnothing \quad \text{and} \quad W_y \cap H_n \neq \varnothing) \\ &\qquad - P(W_0 \cap H_n \neq \varnothing) \cdot P(W_y \cap H_n \neq \varnothing)\}\end{aligned}$$

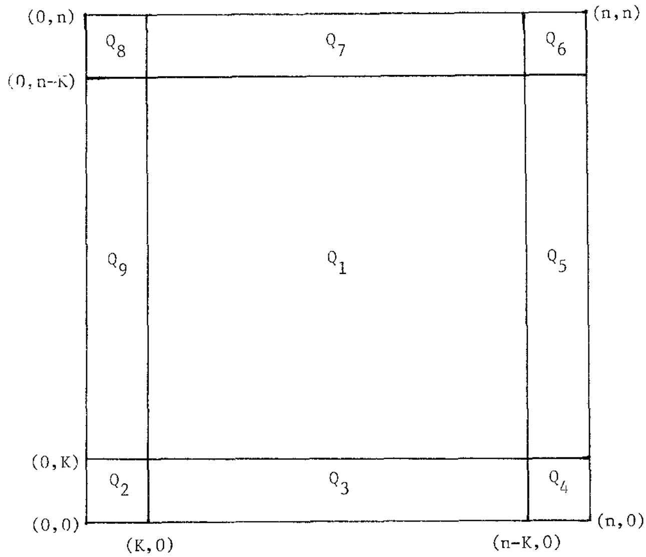

Fig. 1

where $H_n = \{x \in \mathbb{Z}_2 \,|\, x_1 = n\}$. The fact that Λ_1 and Λ_2 are finite follows from Lemmas 1 and 2. We will prove the estimates in (1.3) and omit the similar proof of (1.4).

It is convenient to divide the square as follows (where K is a number to be chosen later) (Fig. 1):

i.e., $Q_1 = \{x \,|\, K \leqslant x_1 \leqslant n - K, K \leqslant x_2 \leqslant n - K\}$, $Q_2 = \{x \,|\, 0 < x_1 < K, 0 < x_2 < K\}$, $Q_3 = \{x \,|\, K \leqslant x_1 \leqslant n - K, 0 < x_2 < K\}$, and so on. Note that $ES_n = \sum_{i=1}^{9} \sum_{x \in Q_i} P(W_x \cap \gamma_n \neq \emptyset)$. We start with Q_1:

$$\sum_{x \in Q_1} P(W_x \cap \gamma(n) \neq \emptyset) \leqslant n^2 \alpha e^{-\beta K} \tag{5.2}$$

For $1 \leqslant i \leqslant K - 1$ define $l_i = \{x \in Q_2 \,|\, \min(x_1, x_2) = i\}$. Then

$$\begin{aligned}\sum_{x \in Q_2} P(W_x \cap \gamma(n) \neq \emptyset) &= \sum_{i=1}^{K-1} \sum_{x \in l_i} P(W_x \cap \gamma(n) \neq \emptyset) \\ &\leqslant \sum_{i=1}^{K-1} |l_i| P(\text{there exists } y \in W_x, d(x, y) \geqslant i) \\ &\leqslant 2\alpha K \sum_{i=1}^{\infty} e^{-\beta i}\end{aligned} \tag{5.3}$$

using Lemma 1. Now let l_0^* be the x axis and for $1 \leqslant i \leqslant K-1$ define $l_i^* = \{x \in Q_3 \mid x_2 = i\}$. Then

$$\sum_{x \in Q_3} P(W_x \cap \gamma(n) \neq \emptyset) = \sum_{i=1}^{K-1} \sum_{x \in l_i^*} \{P(W_x \cap l_0^* \neq \emptyset) + P(W_x \cap l_0^* = \emptyset, W_x \cap \gamma(n) \neq \emptyset)\}$$

$$= \sum_{i=1}^{K-1} (n+1-2K) P(W_0 \cap H_i \neq \emptyset) + \sum_{i=1}^{K-1} \sum_{x \in l_i^*} P(W_x \cap \gamma(n) \neq \emptyset, W_x \cap l_0^* = \emptyset) \tag{5.4}$$

It is not difficult to show that the first term above is $(n-1)\lambda$ plus a term bounded by

$$2K\lambda + n \sum_{i=K}^{\infty} P(W_0 \cap H_i \neq \emptyset) \leqslant 2K\lambda + n \sum_{i=K}^{\infty} \alpha e^{-\beta i} = 2K\lambda + n\alpha e^{-\beta K}/(1-e^{-\beta})$$

The second term is bounded by

$$KnP(\text{there exists } x \in W_0 \quad \text{with} \quad d(0,x) \geqslant K) \leqslant Kn\alpha e^{-\beta K}$$

The terms in Q_4 through Q_9 are similar, and therefore

$$|ES_n - 4(n-1)\lambda| \leqslant n^2 \alpha e^{-\beta K} + 8\alpha K \sum_{i=1}^{\infty} e^{-\beta i} + 8K\lambda + 4n\alpha e^{-\beta K}/(1-e^{-\beta}) + 4Kn\alpha e^{-\beta K}$$

The choice $K = 2\beta^{-1} \ln n$ yields $ES_n = 4(n-1)\lambda + O(\ln n)$. We turn now to the variance estimate

$$\operatorname{Var} S_n = \sum_{x \in \mathring{\gamma}(n)} \left(\sum_{\substack{y \in \mathring{\gamma}(n) \\ d(x,y) \leqslant K}} + \sum_{\substack{y \in \mathring{\gamma}(n) \\ d(x,y) > K}} \right) \{E\xi_n(x)\xi_n(y) - E\xi_n(x)E\xi_n(y)\}$$

$$= \sum_{x \in \mathring{\gamma}(n)} \sum_{\substack{y \in \mathring{\gamma}(n) \\ d(x,y) \leqslant K}} (E\xi_n(x)\xi_n(y) - E\xi_n(x)E\xi_n(y) + O(n^4 e^{-\beta K/4}))$$

using Lemma 2 and the fact that $\xi_{\gamma(n)}$ can be replaced with $\tilde{\xi}_{\gamma(n)}$. We now use the decomposition of $\gamma(n)$.

$$\sum_{x \in Q_1} \sum_{\substack{y \in \mathring{\gamma}(n) \\ d(x,y) \leqslant K}} E\xi_n(x)\xi_n(y) - E\xi_n(x)E\xi_n(y) \leqslant n^2(2K+1)^2 \alpha e^{-\beta K} \tag{5.5}$$

by Lemma 1.

$$\sum_{x \in Q_2} \sum_{\substack{y \in \mathring{\gamma}(n) \\ d(x,y) \leqslant K}} E\xi_n(x)\xi_n(y) - E\xi_n(x)E\xi_n(y)$$

$$= \sum_{i=1}^{K-1} \sum_{x \in l_i} \left(\sum_{\substack{y \in \mathring{\gamma}(n) \\ d(x,y) \leqslant i}} + \sum_{\substack{y \in \mathring{\gamma}(n) \\ i < d(x,y) \leqslant K}} \right)$$

$$\cdot \{ E\xi_n(x)\xi_n(y) - E\xi_n(x)E\xi_n(y) \}$$

$$\leqslant \sum_{i=1}^{\infty} 2K \left\{ (2i+1)^2 \cdot 2\alpha e^{-\beta i} + \sum_{\substack{y \in \mathring{\gamma}(n) \\ d(x,y) \geqslant i}} c_3 e^{-\beta d(x,y)/4} \right\}$$

$$= O(K) \tag{5.6}$$

$$\sum_{x \in Q_3} \sum_{\substack{y \in \mathring{\gamma}(n) \\ d(x,y) \leqslant K}} E\xi_n(x)\xi_n(y) - E\xi_n(x)E\xi_n(y)$$

$$= \sum_{x \in Q_3} \sum_{\substack{y \in \mathring{\gamma}(n) \\ d(x,y) \leqslant K}} P(W_x \cap l_0^* \neq \emptyset, W_y \cap l_0^* \neq \emptyset)$$

$$- P(W_x \cap l_0^* \neq \emptyset) P(W_y \cap l_0^* \neq \emptyset)$$

$$+ \sum_{x \in Q_3} \sum_{\substack{y \in \mathring{\gamma}(n) \\ d(x,y) \leqslant K}} \{ P(W_x \cap \gamma(n) \neq \emptyset,$$

$$W_y \cap \gamma(n) \neq \emptyset, W_x \cap l_0^* = \emptyset \quad \text{or} \quad W_y \cap l_0^* = \emptyset)$$

$$- P(W_x \cap l_0^* \neq \emptyset) \cdot P(W_y \cap l_0^* = \emptyset, W_y \cap \gamma(n) \neq \emptyset)$$

$$- P(W_y \cap \gamma(n) \neq \emptyset) P(W_x \cap l_0^* = \emptyset, W_x \cap \gamma(n) \neq \emptyset) \} \tag{5.7}$$

Since $P(W_x \cap l_0^* = \emptyset, W_x \cap \gamma(n) \neq \emptyset) \leqslant P$ (there exists $y \in W_x, d(x,y) \geqslant K) \leqslant \alpha e^{-\beta K}$, we have

$$\left| \sum_{x \in Q_3} \sum_{\substack{y \in \mathring{\gamma}(n) \\ d(x,y) \leqslant K}} (E\xi_n(x)\xi_n(y) - E\xi_n(x)E\xi_n(y)) \right.$$

$$- \sum_{x \in Q_3} \sum_{\substack{y \in \mathring{\gamma}(n) \\ d(x,y) \leqslant K}} (P(W_x \cap l_0^* \neq \emptyset, W_y \cap l_0^* \neq \emptyset)$$

$$\left. - P(W_x \cap l_0^* \neq \emptyset) P(W_y \cap l_0^* \neq \emptyset)) \right|$$

$$\leqslant 4(2K+1)^2 \cdot K \cdot n\alpha e^{-\beta K}$$

Now we can rewrite

$$\sum_{x\in Q_3}\sum_{\substack{y\in\mathring{\gamma}(n)\\ d(x,y)\leqslant K}} P(W_x\cap l_0^*\neq\varnothing, W_y\cap l_0^*\neq\varnothing)$$
$$-P(W_x\cap l_0^*\neq\varnothing)P(W_y\cap l_0^*\neq\varnothing)$$
$$=\sum_{x\in Q_3}\sum_{\substack{y\in\mathbb{Z}_2\\ y_2>0}} P(W_x\cap l_0^*\neq\varnothing, W_y\cap l_0^*\neq\varnothing)$$
$$-P(W_x\cap l_0^*\neq\varnothing)P(W_y\cap l_0^*\neq\varnothing)$$
$$-\sum_{x\in Q_3}\sum_{\substack{y\in\mathbb{Z}_2\\ y_2>0\\ d(x,y)>K}} P(W_x\cap l_0^*\neq\varnothing, W_y\cap l_0^*\neq\varnothing)$$
$$-P(W_x\cap l_0^*\neq\varnothing)P(W_y\cap l_0^*\neq\varnothing)$$

The first term above is $(n-1)\Lambda_1 - 2(K-1)\Lambda_1$ and it is not difficult to show that the second term is at most $O(nKe^{-\beta K/4})$. Considering the terms from Q_4 to Q_9 we have

$$|\operatorname{Var} S_n - 4(n-1)\Lambda_1| \leqslant O(n^4e^{-\beta K/4}) + 16n^2(2K+1)^2\alpha e^{-\beta K} + O(K)$$
$$+ 2(K-1)\Lambda_1 + O(nKe^{-\beta K/4})$$
$$= O(\ln n)$$

for the choice $K = 16\beta^{-1}\ln n$. ■

NOTE ADDED IN PROOF

We have learned that Gunnar Branvall has independently obtained several central limit theorems similar to ours. He uses different techniques and works with circuits $\gamma(n)$ which are "regular."

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